The quantum Johnson homomorphism, and the symplectic mapping class group of 3-folds

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Plan

Let (M, ω, J) be a Kähler manifold.

- **Gromov-Witten invariants** count the number of isolated holomorphic curves in a given homology class, subject to some generic point constraints.
- Examples:
  - How many conics $(d = 2)$ pass through five generic points? 1.
  - How many lines $(d = 1)$ on a cubic surface? 27.
- The 3-point Gromov-Witten invariants serve as structure coefficients in **quantum cohomology algebra**

$$QH^\bullet(M).$$

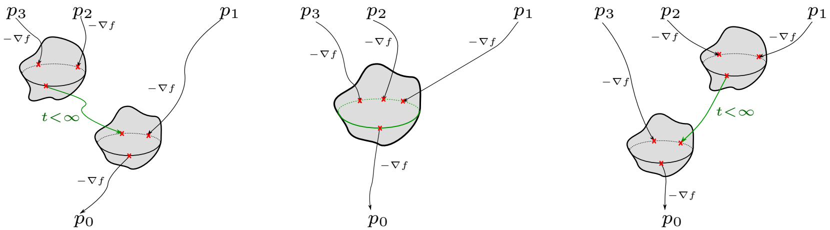
Theorem (K. Fukaya '95, Biran-Cornea '07, FOOO '09,...)
There is an A_∞-refinement

$$(\mathcal{A}, \mu^1 = 0, \mu^2 = \star, \mu^3, \ldots)$$

of the quantum cohomology algebra

$$A := (QH^\bullet(M, \omega), \star).$$

where μ^d is counting **"pearls"** - holomorphic curves connected by Morse flow lines:



- Given a MS pair (f, g), the Morse complex is generated by critical points and has a differential μ^1 and product μ^2.
- There are higher order multilinear operations defined by solving the perturbed gradient equation on *families* of metric trees.

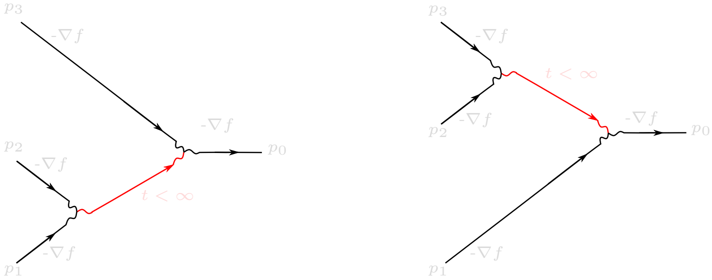


Figure: $\mu^3(p_3, p_2, p_1) = ... + \lambda_{p_0} \cdot p_0 + ...$

Compare : Morse A_∞-algebra

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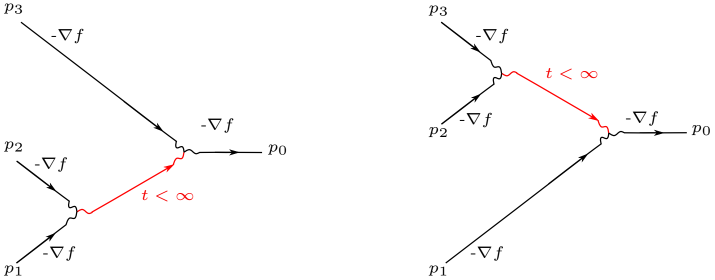

Figure: $\mu^3(p_3, p_2, p_1) = ... + \lambda_{p_0} \cdot p_0 + ...$

Let (M, ω) be a closed symplectic manifold.

- We denote by

$$\mathrm{Symp}(M, \omega) \subset \mathrm{Diff}^+(M)$$

the group of all $\phi : M \to M$ such that $\phi^*\omega = \omega$.

- **The symplectic isotopy problem**: understanding the kernel

$$\pi_0 \mathrm{Symp}(M, \omega) \to \pi_0 \mathrm{Diff}^+(M).$$

- Any symplectomorphism whose isotopy class is in the kernel would be called **exotic**.

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- A symplectic manifold (M, ω) is **monotone** if:

$$c_1(M) = \kappa \cdot [\omega], \ 0 < \kappa \in \mathbb{R}.$$

- **Gromov + Taubes (1985)**:
  $\mathrm{Symp}(S^2 \times S^2, \omega_{mono}) = (SO(3) \times SO(3)) \ltimes \mathbb{Z}_2$,
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- **Abreu, McDuff (1999-2000)**: For the above, $\pi_0 \mathrm{Symp}(M, \omega_{not-mono}) = 0$.
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In dimension 4

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- ...

Seidel: for every other 4dim complete intersection M, there is an **exotic** symplectomorphism τ_V^2.

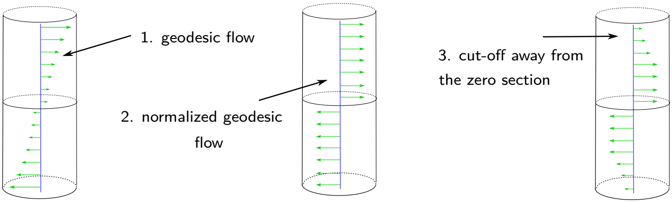

Figure: Model dehn twist in T^*S^n when $n = 1$

J-curves don't behave as nicely (automatic transversality, positivity of intersections ...) in higher dimensions. As a result, very little is known. Some previous results:

- **Seidel (1998)**: Proved the existence of certain π_k ($k > 1$ odd) exotic classes for $M = \mathbb{CP}^n \times \mathbb{CP}^m$ in a range.
- **Ivan Smith (2010)**: Studied $Q_0 \cap Q_1 \subset \mathbb{CP}^5$, the intersection of two quadrics.
- **Weirheim & Woodward (preprint)**: studied the moduli space of flat $SU(2)$-bundles with traceless holonomies on $S^2 \setminus \{\underline{x}\}$.

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The formal package: quantum Johnson homomorphism

- Let X be a closed, monotone, symplectic manifold.
- We associate to every symplectomorphism $\phi : X \to X$ an algebraic "mapping torus"

$$\phi \longrightarrow \mathfrak{X}_\phi := QH^\bullet(X_\phi)$$

and to every symplectic isotopy ϕ_t, a quasi-isomorphism

$$\mathfrak{X}_{\phi_0} \to \mathfrak{X}_{\phi_1}.$$

- Given a homology class $A \in H_2^S(X)$, for suitable ϕ, we assign a "characteristic class"

$$q\tau_2 : \phi \mapsto \mathfrak{X}_\phi \mapsto \mathfrak{o}_\phi^{3,A} \in HH^\bullet(...).$$

which is: Natural w.r.t to quasi-isomorphism; vanishes for $\phi = id$, and can "evaluated" on suitable cohomology classes to give **quantum Massey products**.

Let C_0 be a curve of arithmetic genus $g = 4$ with a single node, which separates C_0 into two irreducible components of genus 2.

- There is a Lefschetz fibration over the unit disc

$$\mathcal{C} \to \Delta$$

  such that:

  1. The special fiber is the nodal curve C_0.
  2. The generic fibers are smooth, non-hyperelliptic curves of genus $g = 4$.
  3. The monodromy ψ_C is symplectically isotopic to a model Dehn twist around a genus 2 vanishing cycle γ.

- There is a relatively very ample line bundle (i.e., the relative canonical bundle) over the punctured unit disc

$$\omega_{\mathcal{C}/\Delta^*}.$$

Family of Fano 3-folds

- The sections of $\omega_{\mathcal{C}/\Delta^*}$ embedd $\mathcal{C} \hookrightarrow \mathbb{P}(\omega_{\mathcal{C}/\Delta^*}) \cong \mathbb{P}^3 \times \Delta^*$.
- Each curve is the complete intersection of a smooth quadric and a cubic.
- We can perform a (complex-analytic) blowup on the family of smooth embedded curves $\mathcal{C} \to \Delta^*$ and get a new family

$$\mathcal{X} \to \Delta^*,$$

with fibers $X_s = Bl_{C_s}\mathbb{P}^3$.
- The further restriction of the base to $S^1 \subset \Delta^*$ is a 7-dim **Locally Hamiltonian fibration**.
- Thus, the monodromy obtained by parallel transport around the circle

$$\phi : X \to X$$

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Pick a base point on $* \in S^1$ and denote $X := X_*$.

- **Main result (B.)** The symplectic isotopy class $\kappa = [\phi]$ has infinite order in the kernel

$$\pi_0 \mathrm{Symp}(X, \omega_{mono}) \to \pi_0 \mathrm{Diff}^+(X)$$

- In fact, there exists Lagrangian spheres $\{V_1', \cdots, V_5'\}$, and $\{V_1'', \cdots, V_5''\}$ in X such that κ has a factorization in the symplectic mapping class group as the product of generalized Dehn twists about them!
- Note: Unlike 4dim, Dehn twist in odd dimensions can have infinite order in $\pi_0 \mathrm{Diff}^+(X)$!

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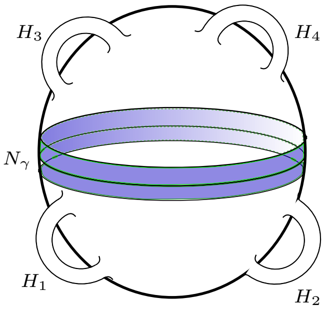
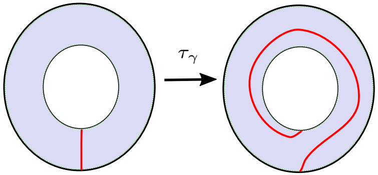
Let C be a genus 4 curve. How can we prove that the *seperating Dehn twist* $\psi = \tau_\gamma$ is not isotopic to the identity?



(a) A tubular neighbourhood

$$A \cong N_\gamma \hookrightarrow C$$



(b) Effect of model Dehn twist

$$(s, t) \mapsto (se^{\mathbf{i}t}, t)$$

The Johnson homomorphism τ_2

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- Consider the mapping torus

$$C_\psi := \mathbb{R} \times C / \sim \ , \ (t, x) \sim (t-1, \psi(x)).$$

- Fact: on the level of *cohomology algebras*

$$H^\bullet(C_\psi) \cong H^\bullet(C \times S^1) \cong Q[\mathfrak{t}]/(\mathfrak{t}^2),$$

  where $Q^\bullet = H^\bullet(C)$ and $\mathfrak{t}$ is a formal parameter of degree 1.

- D. Johnson (1980): the triple Massey products

$$Q^1 \otimes Q^1 \otimes Q^1 \to \mathfrak{t} \cdot Q^1 \subset H^2(C_\psi)$$

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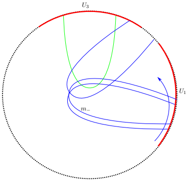
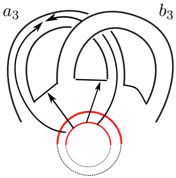
Non-formality of C_ψ ... or picture proof for $\tau_2(\psi) \neq 0$

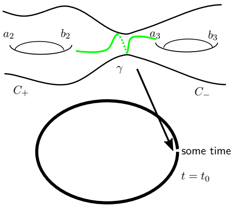
The negative gradient flow of a generic point in C_+ **passes once through every point** in C_-.



(a) "Rotating cap" has no influence



(b) New differentials cancel out



(c) ...A new μ^3 appears

Claim

$$\left\langle \begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} b_1 & 0 \\ -b_2 & 0 \end{pmatrix}, \begin{pmatrix} b_1 & b_3 \\ 0 & 0 \end{pmatrix} \right\rangle = \begin{pmatrix} 0 & \mathfrak{t}b_3 \\ 0 & 0 \end{pmatrix}$$

is a nontrivial coset (hence $\mathfrak{C}$ is non-formal).

(Classical) Massey products in symplectic blowups

Let Y, X be closed symplectic manifolds, and $Y \hookrightarrow X$ a symplectic embedding of codimension $2r$.

- **RTBT theorem (1999)**. Assume that the codimension $2r$ is ≥ 8. Then Y has a nontrivial Massey product $\Rightarrow Bl_Y X$ also has a nontrivial Massey product.
- This was used in the construction of some early examples of non-Kähler symplectic manifolds.
- Contrast with **DGMS (1975)**: Let X be a compact Kähler manifold. Then the de-Rham algebra $(\Omega^\bullet(X), d)$ is formal.
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Where does "exotic" come from?

What happens after the blowup?

- Unfortunately, all the information stored in the triple Massey product is **lost** (to classical topology).
- The cohomology of the blowup is (additively)

$$H^*(Bl_C\mathbb{P}^3) = H^*(\mathbb{P}^3) \oplus H^*(C)[u]$$

where $|u| = 2$ and $u^3 = 0$, and

$$H^3 \otimes H^3 \otimes H^3 \to H^{3+3+3-1} = H^8 = 0.$$

- **It's not a bug - it's a feature!** The isotopy class of the monodromy ϕ is controlled by topological data (rational homotopy + pontryagin classes) up to "finite ambiguity".
- **Idea.** Unlike most cohomology theories, quantum cohomology is **not nilpotent**, e.g., $QH^\bullet(\mathbb{CP}^n) = \mathbb{Z}[h,q]/(h^{n+1} = q)$.

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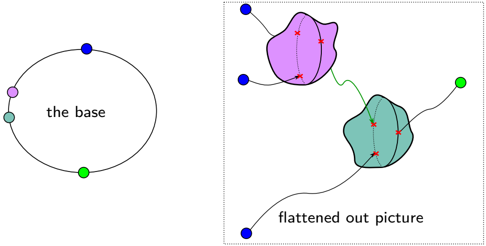
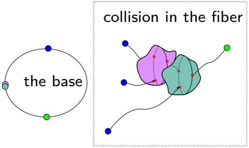
To define quantum Massey products we need a *parametrized version* of the quantum A_∞-structure.

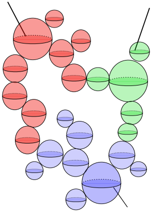
- We consider the symplectic fibration $X_\phi \to S^1$.
- Morse trajectories are allowed to move in the total space X_ϕ.
- J-Holomorphic curves live in the fibers (X_s, ω_s).



- A multiple cover type-problem



- Solution: need to let perturbations depend on the simultaneous position of all pearls.



Assume that there is an associative product structure $A = (\mathcal{A}, \cdot)$.

Definition

Let $\mathfrak{U}(A)$ be the set of A_∞-structures μ on the underlying vector space of $\mathcal{A}$ with $\mu^1 = 0$ and $\mu^2(x_2, x_1) = (-1)^{|x_1|} x_2 \cdot x_1$. There is a group of *gauge transformations*

$$\mathfrak{G}(A) := \left\{ \mathcal{G}^1 = id, \mathcal{G}^2 : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}[-1], \dots \right\}$$

which acts on $\mathfrak{U}(A)$.

Idea. Think of μ as a *deformation* of the **formal** A_∞-structure $(0, \cdot, 0, 0, \dots)$.

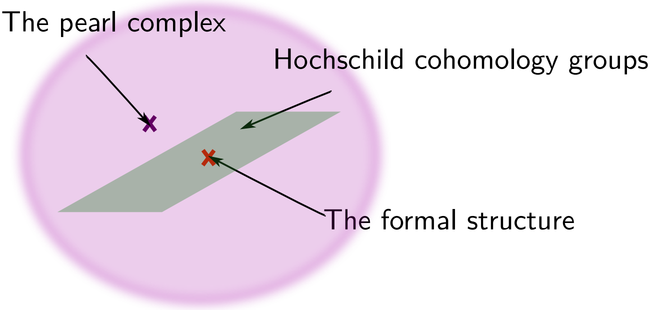
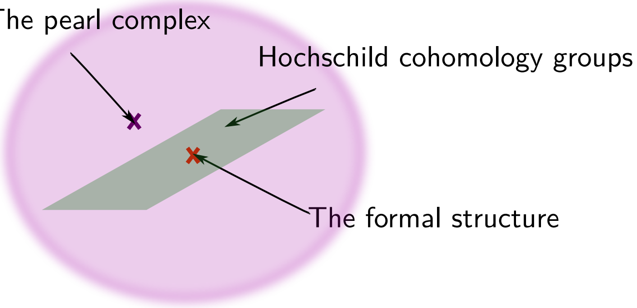


Figure: The space of minimal A_∞-structures $\mathfrak{U}(A)/\mathfrak{G}(A)$



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The "angle of arrival" of μ^d under the $\mathbb{C}^*$-action

$$\mu^d \mapsto \epsilon^{d-2}\mu^d \; , \; \epsilon \neq 0$$

gives an obstruction class

$$\mathfrak{o}^3 = [\mu^3] \in HH^2(\mathcal{A})^{-1}.$$

in the "tangent space" $HH^2(A, A)^{-1}$.

Let $[a] \in A$ be a cohomology class. We want to define a way to "evaluate" $\mathfrak{o}^3$ on cohomology classes.

- Quantum Massey products are partially defined multilinear operations

$$\langle [x_3], [x_2], [x_1] \rangle_{[a]} : A^p \otimes A^q \otimes A^r \to A^{p+q+r-1-2c_1([a])}$$

which depend only on the homology class $[a]$, the inputs $[x_3], [x_2], [x_1]$ and $\mathfrak{o}^3 = [\mu^3]$.

- **Vanishing condition**. We must have

$$[x_2] \star_{[b]} [x_1] = [x_3] \star_{[b]} [x_2] = 0 \,,\ \forall [b] \in A^2 \text{ s.t. } c_1([b]) \leq c_1([a]).$$

- **Ambiguity**. The ideal I is generated by

$$A \star_{[b_2]} [x_1] + [x_3] \star_{[b_1]} A$$

for all $c_1([b_1]), c_1([b_2]) \leq c_1([a])$.

- Choose **bounding cochains**, i.e.,

$$h = \sum h_B \cdot q^B \ , \ g = \sum g_B \cdot q^B$$

such that $\mu^1(h) = \mu^2(x_3, x_2)$ and $\mu^1(g) = \mu^2(x_2, x_1)$.

- **Definition**. We set

$$\langle x_3, x_2, x_1 \rangle_{[a]} = [\mu^3_{[a]}(x_3, x_2, x_1) \pm \sum_{[b_1]+[b_2]=[a]} \mu^2_{[b_1]}(x_3, h_{[b_2]}) \pm \sum_{[b_1]+[b_2]=[a]} \mu^2_{[b_1]}(h_{[b_1]}, x_1)]/I.$$

- **Key point:** sort of "Finite determinicity" makes computations tractable (especially when using an extermal ray as the homology class).

- The effective cone is generated by F, and R.

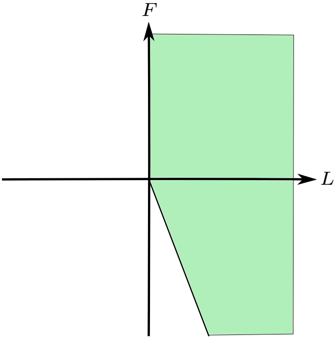

Figure: The cone of curves

- $c_1 = 1$: F (the exceptional fiber), and $R = L - 3F$ (ruling lines).
- $c_1 = 2$: $2F$, $L - 2F$ (lines not in Q) and $2L - 6F$ (conics).

The "Watchtower correspondence"

The first step is to extend $\mathcal{X} \to \Delta^*$ over the zero fiber and write the whole family as a Kähler degeneration with smooth total space and ADE-singularities in the central fiber.

- **Problem:** the sections of the dualizing sheaf **fails to embed** curves with genus 2 tails like $C_0 \in \delta_2 \subset \overline{\mathcal{M}}_4$ in $\mathbb{P}^3$.
- **Solution:** Consider the homogeneous cubic form

$$f = X_0 X_3^2 + X_1^2 X_4 - X_0 X_2 X_4 + X_1 X_2 X_3.$$

  The zero set in a cubic 3-fold $F_{2A_5} := V(f) \subset \mathbb{P}^4$.
- F_{2A_5} has a ODP at $p = [0:0:1:0:0]$ and two A_5-singularities.

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The "Watchtower correspondence"

Given any cubic hypersurface $F = V(f) \subset \mathbb{P}^4$ with ODP at $p = [0:0:1:0:0]$ as above, we can write

$$f = X_2 \cdot q + s$$

with q and s homogeneous of degrees 2 and 3 respectively.

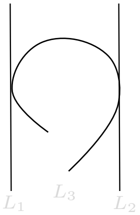
- Consider the projection

$$\pi_p : F \to V(X_2) = \mathbb{P}^3.$$

- We can resolve π_p by blow up. Denote $X = Bl_p F$.
- The exceptional locus is the proper transform of lines via p that lie in F, which is the cone over $C := V(q, f)$. Thus

$$Bl_C \mathbb{P}^3 = X = Bl_p F$$

- **Theorem (Wall '98).** Let us fix a singular point $x \in C$, and denote $\ell = \overline{px}$. F has exactly two singular points p, q on ℓ. *The type of q is the same as that of x.*
- In our case, the complete intersection $C_{2A_5} := V(q, s) \subset \mathbb{P}^3$ is the curve



where L_1 and L_2 are $(1,0)$-ruling lines of the quadric $Q = V(q)$, and L_3 is a smooth rational $(1,3)$-curve.

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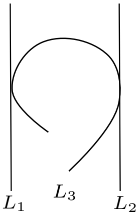

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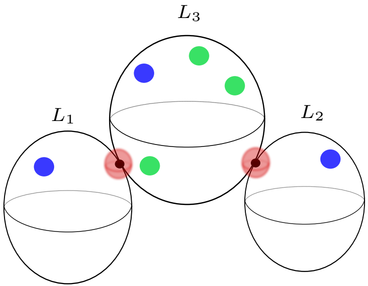
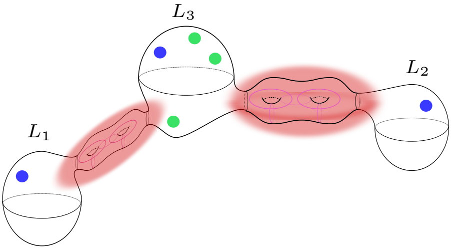
We have a new construction of the S^1-family – by smoothing C_{2A_5}.



(a) The singular fiber



(b) A generic fiber

Short summary

- As evinced by D. Johnson's work, the rational homotopy theory of mapping tori of $\psi : C \to C$ is very rich, and can detect elements of the mapping class group efficiently.
- Degree changing operations (like blowup/branched covers/moduli of bundles...) can kill the Massey products which leads to "unknottedness" of $\phi : X \to X$.
- Symplectic invariants are not nilpotent and can still carry interesting information.
- Computing quantum Massey products of low energy allows us to prove the non-triviality of the symplectic isotopy class.

Questions?